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MATHEMATICAL MODELING OF PENETRATION OF A MIXED LAYER INTO A STRATIFIED FLUID

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The paper deals with the mathematical formulation and numerical solution of the two-dimensional problem of penetration of a turbulent mixed layer into a linearly stratified fluid under the action of a friction stress. The $(e - \varepsilon)$ model containing equations for the turbulence energy and its dissipation rate is used in turbulence simulation. We study numerically the behavior of the solution of the problem with various values of the buoyancy-force parameter in the equation for the dissipation rate of turbulence energy. The calculation results for the thickness of the mixed layer are in good agreement with the experimental data obtained by Kato and Phillips [1].

During warm seasons, deep reservoirs are usually stratified. The upper layers of water in them are subject to mixing due to turbulence- and convection-induced processes. The processes of turbulent mixing in the upper layers of deep reservoirs play an important part in the formation of their thermal structure, in the appearance of a thermocline. The thermocline is a layer that prevents the transfer of oxygen and nutrient substances in reservoirs and, hence, exerts a significant influence on functioning of water ecosystems. One of the main mechanisms that generate the upper mixed layer, namely, turbulence generation due to a tangential stress applied to the water surface and development of the turbulent motion in a steadily stratified fluid, was studied experimentally in [1]. The processes of turbulent mixing were simulated numerically in [2–4] under conditions of the laboratory experiment of [1] under the assumption on the uniformity of hydrodynamic parameters along the length of the trough. In [5], an attempt was made to eliminate the restriction on the uniformity of these parameters along the length of a trough, but only for a closed trough under the no-slip condition at its ends.

In the present paper, we propose a problem that reflects the flow in a circular trough more precisely. At the end cross sections, we impose the free-flow condition at the outlet boundary and also the condition that the flow at the inlet boundary is an uninterrupted continuation of the flow at the exit from the considered domain of solution of the problem. The processes of vertical turbulent mixing are described on the basis of the complete $(e - \varepsilon)$ model, while the coefficients of horizontal turbulent exchange are determined by the Richardson formula. Moreover, this formulation of the problem takes into account the friction stress of the side walls. The possibility of the effect of these walls on the mixed-layer thickness was mentioned in [6].

Formulation of the Problem. Water flows in a narrow long trough under the laboratory conditions described in [1]. Therefore, one can apply an approach that allows the initial equations of motion and transport of salt over the trough width in the radial direction to be averaged [7]. The equations of momentum and mass conservation and the equation for the density of salt water are then of the form [7, 8]

$$\frac{\partial bu_1}{\partial t}+\frac{\partial bu_1u_1}{\partial x_1}+\frac{\partial bu_1u_2}{\partial x_2}=-gb\frac{\partial}{\partial x_1}\Big(z+\frac{1}{\rho_0}\int\limits_{x_2}^{z}\rho\,dx\Big)+\frac{\partial}{\partial x_1}bK_h\frac{\partial u_1}{\partial x_1}+\frac{\partial}{\partial x_2}bK_v\frac{\partial u_1}{\partial x_2}-2r|u_1|u_1; \qquad (1)$$

$$\frac{\partial bu_1}{\partial x_1}+\frac{\partial bu_2}{\partial x_2}=0; \qquad (2)$$

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$$\frac{\partial b\rho}{\partial t} + \frac{\partial b u_1 \rho}{\partial x_1} + \frac{\partial b u_2 \rho}{\partial x_2} = \frac{\partial}{\partial x_1} b K_{hs} \frac{\partial \rho}{\partial x_1} + \frac{\partial}{\partial x_2} b K_{vs} \frac{\partial \rho}{\partial x_2}, \tag{3}$$

where t is the time, x_1 and x_2 are the axes of the Cartesian coordinate system (the x_2 axis is directed vertically upright); u_1 and u_2 are the velocity components along x_1 and x_2, respectively; ρ is the density of the aqueous salt solution; b is the trough width; g is the acceleration of gravity; z is the level of the water surface; K_h and K_v (K_{hs} and K_{vs}) are the coefficients of total (turbulent and molecular) viscosity (salt diffusion) in the horizontal h and vertical v directions, respectively; and r is the coefficient of friction resistance of the walls.

It should be noted that the density equation (3) follows from the diffusion equation of salt transport under the assumption of a linear dependence between the solution density and the salt concentration. Moreover, the coefficients of turbulent heat and salt transport are assumed to be equal.

The coefficients of vertical turbulent exchange are determined using the equations for turbulence energy e and dissipation rate ε [9, 10]:

$$\frac{\partial be}{\partial t} + \frac{\partial b u_1 e}{\partial x_1} + \frac{\partial b u_2 e}{\partial x_2} = \frac{\partial}{\partial x_1} b K_{he} \frac{\partial e}{\partial x_1} + \frac{\partial}{\partial x_2} b K_{ve} \frac{\partial e}{\partial x_2} + b(P - G) - b\varepsilon; \tag{4}$$

$$\frac{\partial b\varepsilon}{\partial t} + \frac{\partial b u_1 \varepsilon}{\partial x_1} + \frac{\partial b u_2 \varepsilon}{\partial x_2} = \frac{\partial}{\partial x_1} b K_{h\varepsilon} \frac{\partial \varepsilon}{\partial x_1} + \frac{\partial}{\partial x_2} b K_{v\varepsilon} \frac{\partial \varepsilon}{\partial x_2} + c_{1\varepsilon} \frac{\varepsilon}{e} b(P - (1 - c_{3\varepsilon})G) - c_{2\varepsilon} b \frac{\varepsilon^2}{e}. \tag{5}$$

Here K_{he} ($K_{h\varepsilon}$) and K_{ve} ($K_{v\varepsilon}$) are the coefficients of total diffusion of turbulence energy (dissipation rate) in the horizontal h and vertical v directions; $P = K(\partial u_1/\partial x_2)^2$; $G = -g(\alpha_s/\rho_0)K(\partial\rho/\partial z)$; the coefficients of turbulent exchange are found by the following formulas [9, 10]: $K = c_\mu e^2/\varepsilon$, $K_v = \nu + K$, $K_{vs} = \nu_s + \alpha_s K$, $K_{ve} = \nu + \alpha_e K$, and $K_{v\varepsilon} = \nu + \alpha_\varepsilon$; ν and ν_s are the molecular viscosity and the salt diffusivity; $c_\mu = 0.09$, $\alpha_s = 0.8$; $\alpha_e = 1.0$; and $\alpha_\varepsilon = 0.77$. We use the standard values of the remaining constants: $c_{1\varepsilon} = 1.44$, $c_{2\varepsilon} = 2.0[1.0 - 0.3\exp(-Re_T^2)]$, $Re_T = e^2/(\nu\varepsilon)$, and $c_{3\varepsilon} = 0.8$. The total coefficients of horizontal exchange K_h, K_{hs}, K_{he}, and $K_{h\varepsilon}$ are assumed to be constant. Their numerical values are given below.

For system (1)–(5), the following boundary conditions are adopted:

— at the ends ($x = x_L$ is the left cross section through which water flows into the region under consideration, and x_R is the right cross section through which water flows out)

$$\varphi(t, x_L, x_2) = \varphi(t, x_R, x_2) \quad [\varphi = (u_1, \rho, e, \varepsilon)]; \tag{6}$$

$$\partial\varphi/\partial x_1 = 0 \quad \text{for} \quad x_1 = x_R; \tag{7}$$

— on the water surface for $x_2 = z$

$$K_v \frac{\partial u_1}{\partial x_2} = \frac{\tau_w}{\rho}, \quad \frac{\partial \rho}{\partial x_2} = 0, \quad \frac{\partial e}{\partial x_2} = 0, \quad \varepsilon = c_\varepsilon \frac{e^{3/2}}{y^0}; \tag{8}$$

— at the bottom for $x_2 = 0$

$$K_v \frac{\partial u_1}{\partial x_2} = k_b |u_1| u_1, \quad \frac{\partial \rho}{\partial x_2} = 0, \quad u_2 = 0, \quad \frac{\partial e}{\partial x_2} = 0, \quad \varepsilon = c_\varepsilon \frac{e^{3/2}}{y_0}. \tag{9}$$

Here y^0 and y_0 are the roughnesses of the water surface and of the bottom respectively, $k_b = 0.14$, and $c_\varepsilon = 0.314$ [11].

It should be noted that in this physical experiment a constant tangential stress τ_w was applied to the water surface by displacement of the cover of the circular trough. Therefore, the level of the water surface is assumed to be constant in the problem. Since the water flow is studied in a circular trough [1], the left and right end cross sections in the flow pattern correspond to the same arbitrary vertical cross section of the trough. Condition (7) means free liquid flow at the outlet section, while (6) implies an uninterrupted continuation of flow at the inlet section.

In addition to the boundary conditions (6)–(9), for system (1)–(5) one should specify the initial conditions corresponding to the state of rest.

Calculation Results. A numerical solution of the problem stated is constructed on the staggered Arakawa C-grid [12]. In this case, the scalar values ρ, e, and ε are determined at the central points of the

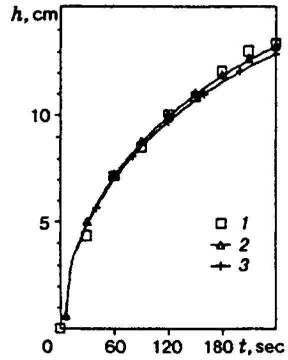

Fig. 1

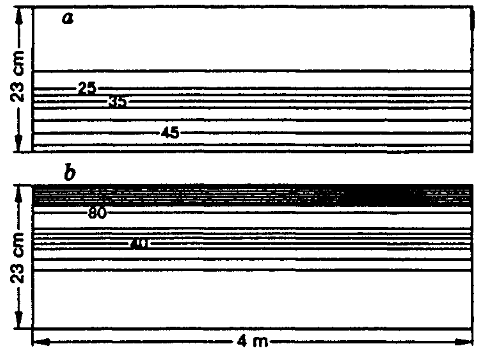

Fig. 2

grid, and the velocity components are determined at the boundaries of the grid elements. The approximation method for source terms in Eqs. (4) and (5) with its test by a simple example of flow is presented in [11]. The horizontal velocity component is calculated using the splitting method [13]. In the first fractional step, variation in the horizontal momentum due to advection and diffusion is calculated at the central points with the pressure forces ignored. In the second fractional step, the velocity fields are adapted to the pressure distribution at the edges of the grid elements. Recall that the pressure distribution is considered hydrostatic and is, therefore, determined by the density distribution (if the level is constant). Then, the terms of horizontal advection and horizontal diffusion and also the vertical coefficients are calculated explicitly (from the known layer in time). We also used the sweep method in the vertical direction.

Recall that the laboratory experiments were conducted under a constant tangential stress near the water surface directly under the trough cover. At the initial moment, salt water was at rest and had a linear density distribution over the depth. For simulation, we performed calculations for the density gradient $\partial\rho/\partial x_2 = 1.92 \cdot 10^{-3}$ g/cm^4, friction stress at the trough cover $\tau_w = 0.995$ dyn/cm^2, water depth $h = 0.23$ m, and trough length 4 m. The values of the total horizontal-exchange coefficients of K_h, K_{hs}, K_{he}, and $K_{h\epsilon}$ were assumed to be equal and were calculated using the Richardson formula $K_h = cl^{4/3}$, where the value of l was accepted equal to the grid size in the horizontal direction (subgrid-turbulence parametrization), the numerical coefficient was $c = 0.01$ cm$^{2/3}$/sec [14], $K_h = 5.6 \cdot 10^{-5}$ m^2/sec, the time step was equal to 0.5 sec, the number of points of the grid over the trough depth and length was 40 and 20, respectively, and $r = 1.32 \cdot 10^{-3}$. A twofold decrease in the time step had practically no effect on the calculation results.

Figure 1 shows the measurement [1] (points 1) and calculation results for the change in the mixed layer thickness at the center of the trough for $c_{3\epsilon} = 0.8$ [10] and 1.0 [15] (curves 2 and 3) in the equation for turbulence-energy dissipation. For $c_{3\epsilon} = 0.8$ and 1.0, the curve shows good agreement with the experimental data and differ slightly from one another, but the calculation for $c_{3\epsilon} = 0.8$ gives a more preferable result.

Figure 2a shows the density-difference distributions $\rho - \rho_0$ (ρ, kg/m^3 and $\rho_0 = 1000$ kg/m^3), and Fig. 2b shows the ratios of the coefficients of turbulent and molecular viscosity in the vertical plane at time $t = 240$ sec for $c_{3\epsilon} = 0.8$. All isolines are equidistant. The behavior of the density isolines and the turbulent-viscosity coefficient along the length of the trough exhibits their uniformity over the length.

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